

# Generating conditional atomic entanglement by measuring photon number in a single output channel

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## Abstract

The polarization analysis of quantized probe light transmitted through an atomic ensemble has been used to prepare entangled collective atomic states. In a "balanced" detection configuration, where the difference signal from two detection ports is analyzed, the continuous monitoring of a component of the Stokes field vector provides a means for conditional projective measurements on the atomic system. Here, we make use of classical driving fields, in the pulsed regime, and of an "unbalanced" detection setup (single detector) where the effective photon number of scattered photons is the detected observable. Conditional atomic spin squeezed states and superpositions of such squeezed states can be prepared in this manner.

## I. INTRODUCTION

The generation of entanglement in an atomic ensemble of macroscopic dimensions is a challenging task in the fields of quantum information, quantum teleportation and precision measurements. Out of the multitude of the entangled states that could conceivably be engineered using quantum optics techniques, two subsets have attracted a special interest: spin squeezed states and "Schrodinger cat" states. The first subset is needed for improving the precision of quantum measurements beyond the standard quantum limit, while the second one provides an example of a purely quantum mechanical superposition at the macroscopic level. The concept of spin squeezing has been introduced in a comprehensive manner in [1] which also provide the first theoretical proposals for its realization. Much attention has been given to the generation of atomic squeezing in the context of the interaction of cavity quantized fields with atoms, resulting in a transfer of squeezing to the atoms from a field initially in a squeezed state [2], or even from an initially coherent field state [3]. Free space squeezing transfer has also been predicted [4] and analyzed experimentally [5].

More recent proposals for the generation of entangled atomic states are based on the dispersive effect of off-resonant light-atom interactions. Information about the atomic state is correlated with the phase shift accumulated by the field, and can be "read" by performing a suitable measurement of the field. The reverse effect of measuring a phase shift acquired by Rydberg atoms in passing through a microcavity has already been used to generate a "Schrodinger cat" state of a quantized field in a cavity [6]. References [7, 8] provide other examples of phase-shift measurements that lead to a spin squeezed atomic state and entanglement between two macroscopic atomic ensembles. It has been shown [9] that, with a suitably chosen internal atomic configuration and quantized fields, for the off-resonant regime, the interaction can be modelled in the form of a quantum non-demolition (QND) Hamiltonian proportional to  $S_z^a J_z^f$ , where  $S_z^a$  represents the signal observable (atomic population spin component) while  $J_z^f$  (component of the Stokes vector of the field) is the probe observable to be measured. Most often, a continuous measurement treatment of the problem is employed, where the polarization analysis of the probe light transmitted through the medium is detected in a "balanced" configuration with two photodetectors [10, 12]. In [11, 12], sub-shot-noise fluctuations and spin squeezing with 70% reduction below the standard quantum limit have been achieved. The continuous character of the probe monitoring

has also rendered possible an implementation of a feed-back loop for *a posteriori* quantum state correction based on the conditional measurement of the outcome [13, 14].

In this paper a different approach is adopted to analyze the manner in which entanglement can be generated by means of field detection. In contrast to other theories, we consider the input field to be a classical field, rather than a (quantized) coherent state of the field. This is consistent with most experiments involving large scale entanglement, where the input field is a laser field, and quantum fluctuations of this field play a negligible role. The quantum properties of the output field are then related to the radiation scattered by atoms interacting with the input field.

We apply this treatment to an atomic medium of 4-level Faraday active atoms ( $J_g = 1/2 \rightarrow J_e = 1/2$ ) that can rotate the direction of polarization of an incoming field, given a population imbalance between the ground magnetic sublevels. One prepares the atomic ensemble in a superposition of ground state spin up and spin down with an equal admixture of both states. In this case, on average, there is no Faraday effect. Faraday rotation is induced solely by quantum fluctuations in the population difference of the spin up and spin down states. With an initially  $x$  polarized driving field, the Faraday effect can be seen as due to a redistribution of photons between the  $x$  and  $y$  polarizations of the field. The  $y$  polarized part is simply the source field emitted by the atoms and thus entangled with the atomic state; in consequence, a measurement of this Faraday rotated part of the field allows gathering of information on the atomic ensemble with a measurement strength proportional to the laser power (as shown in [15]).

Our approach also differs from other approaches in that we chose a different "unravelling" of the dynamics of the system. This is done by the choice of the measurement basis that we consider, which is given by an "unbalanced" detection setup with a polarization beam splitter used to separate the  $x$  polarized light (directed into an unmonitored port) from the  $y$  polarized part that is directed into a photodetector. Owing to the pulsed regime considered here, the continuous character of the detection is also lost, leading to a formulation of the measurement as a "discrete" detection of the total photon number in the scattered pulse. Conditioned on the outcome of such a measurement, one can guarantee that the atomic ensemble is in a spin squeezed state or a coherent superposition of squeezed states ("Schrodinger cat" state). Only a single operation is necessary here to prepare a bimodal squeezed state ("Schrodinger cat" state), as opposed to the procedure of Ref.[16] which

requires two QND steps to prepare similar cat states. Intuitively, the reduction from two steps to only one comes from the indistinguishability of the two possible rotations (left-right) of light corresponding to a given measured number of scattered photons.

The main results of the paper are obtained in the ideal case where spontaneous decay to modes other than the optical mode of interest is neglected; the unavoidable limitations imposed by decay are discussed in Section IV and are consistent with the general conclusion [7, 16] that the resonant optical depth of the atomic ensemble sets the limit on the optimal achievable entanglement for the collective atomic state. However, we find the ideal case conclusions to be very useful in terms of providing a clear physical picture of entanglement generation. Therefore, even if our measurement strength parameter  $C$  is limited by spontaneous decay to values less than unity, we present results for large values of  $C$  to illustrate the atom-field entanglement. It should be also noted that other limitations will come into play when a realistic evaluation of the proposed setup is made, such as non-ideal of polarizers, and photodetection dark counts.

The paper is organized as follows. The proposed experimental scheme and some fundamental notions about spin squeezing are introduced in Section II. Section III is devoted to the study of the Faraday effect for a single atom and for many atoms. The single atom scattered field is shown to reflect atomic population fluctuations, while collective effects in the scattering process occur for the many atom case. In Sec. IV the effective Hamiltonian needed to describe the generation of the source field from atomic fluctuations is derived; the spontaneous emission issue is also addressed here. Also, the back-action on the atomic state of a measurement of  $n$  photons in the field is derived using a quantum trajectories method. A simple interpretation of the photon statistics of the scattered field and the connection between measurement outcomes and projected collective atomic states, is given in Sec. V. Conclusions are presented in Sec. VI and some detailed calculations for Sections III and IV are carried out in the Appendices.

## II. ATOM-FIELD SYSTEM

An  $x$  polarized classical laser pulse (carrier wavelength  $\lambda$ ) propagating in the  $z$  direction [Fig.1(a)] drives off-resonantly a  $J_g = 1/2 \rightarrow J_e = 1/2$  atomic transition [Fig.1(b)]. Assuming non-interacting, stationary atoms (number of atoms  $N_a$  and atomic density  $n_a$ ) in a

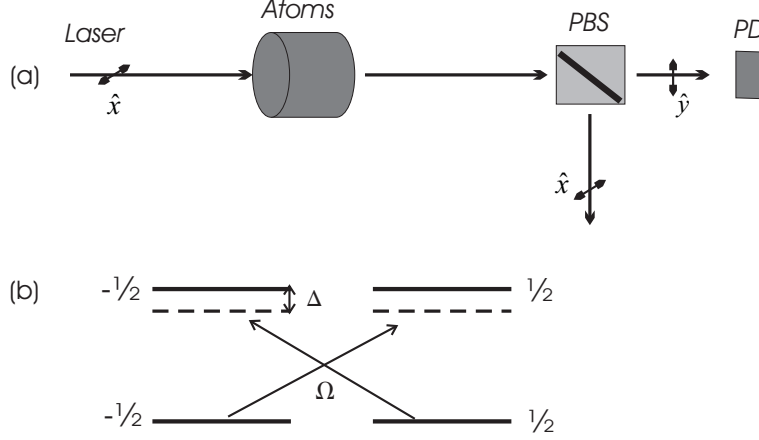


FIG. 1: (a) An  $x$  polarized classical laser pulse passes through a medium of Faraday active atoms, suffering a rotation of polarization, i.e. a  $y$  polarized field is created. This scattered field is filtered at the PBS (polarization beam splitter) and detected at the PD (photodetector). (b) internal structure of an individual atom in the Faraday active medium.

pencil-shaped medium with transverse area  $A$  (matched to fit the pulse area), length  $L$  and Fresnel number close to unity ( $F = \frac{A}{\lambda L} \approx 1$ ), the scattered field intensity is confined mainly to the forward direction [17]. With conveniently chosen parameters, the resonant optical depth ( $d_{res} = n_a \lambda^2 L$ ) can be made large, while the off-resonant depth is kept small to ensure a small saturation of the atomic medium (negligible field depletion). A population imbalance between the ground state sublevels induces a rotation of the overall polarization of the field (Faraday effect); in other words, the emerging field (after the interaction) consists of an unscattered part (whose intensity is equal to that of the incoming pulse since the depletion of the field due to absorption is neglected), an  $x$  polarized scattered quantized field, and a  $y$  polarized scattered quantized field. The ratio of the  $y$  polarized field amplitude over the incoming field amplitude defines the Faraday rotation angle. The  $y$  component of the emerging field is entangled with the atoms. If a PBS (polarization beam splitter) is used to filter out the  $x$  polarized field (which is not entangled with the atoms), a measurement at the photodetector (PD) will generate entanglement within the atomic ensemble.

Single atom population operators and coherences that are used in what follows are defined as  $\sigma_g(m_g, m'_g) = |g, m_g\rangle \langle g, m'_g|$  with  $m_g, m'_g = \pm 1/2$  (for the ground state manifold),  $\sigma_e(m_e, m'_e) = |e, m_e\rangle \langle e, m'_e|$  with  $m_e, m'_e = \pm 1/2$  (excited state manifold) and  $\sigma_+(m_e, m_g) =$

$|e, m_e\rangle \langle g, m_g|$  for  $m_g, m_e = \pm 1/2$  (raising operator), along with its hermitian conjugate  $\sigma_- (m_e, m_g) = [\sigma_+ (m_e, m_g)]^\dagger$ . For the two levels of interest (the ground magnetic sublevels) collective spin operators are written as sums over single atom operators

$$\begin{aligned} S_z &= \sum_{j=1}^N \left[ \sigma_g^{(j)} \left( \frac{1}{2}, \frac{1}{2} \right) - \sigma_g^{(j)} \left( -\frac{1}{2}, -\frac{1}{2} \right) \right] / 2 \\ S_+ &= \sum_{j=1}^N \sigma_g^{(j)} \left( \frac{1}{2}, -\frac{1}{2} \right) \\ S_- &= S_+^\dagger \\ S_x &= (S_+ + S_-) / 2 \\ S_y &= (S_+ - S_-) / 2i \end{aligned} \tag{1}$$

The initial state of the collection of atoms is chosen as an eigenstate of  $S_x$  with equal populations in the two ground sublevels and initial maximal built-in coherence. This state can be expanded in eigenstates of  $S_z$  as

$$|S_x = S\rangle = \sum_{M=-S}^S A(S, M) |S, M\rangle \tag{2}$$

Here  $A(S, M) = \frac{1}{2^S} \sqrt{\frac{(2S)!}{(S+M)!(S-M)!}}$  are binomial coefficients, while  $S = N_a/2$ . A general squeezing parameter is defined as:

$$\xi_\perp = \sqrt{2S} \Delta S_\perp / |\langle \mathbf{S} \rangle| \tag{3}$$

where  $|\langle \mathbf{S} \rangle| = \sqrt{\langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2}$  is the instantaneous mean spin and  $S_\perp$  is a component of the spin orthogonal to the mean spin vector. An atomic system is said to be spin squeezed when this parameter takes subunitary values. For the state described in Eq. (2), a squeezed state can be obtained by minimizing the uncertainty in  $S_z$  (generating a sub-binomial distribution), while keeping the average spin in the  $x$  direction. A fundamental quantum limit (Heisenberg limit) for  $\xi_\perp$  exists and equals  $1/\sqrt{N_a}$  [1].

### III. FARADAY EFFECT

In this section we study the properties of the source field radiated by an ensemble of four-level atoms [Fig.1(b)] driven by a classical field. A single atom treatment is shown to be sufficient to describe the generation of a Faraday rotation. Some results on the amplitude

and intensity of the source field and on its origin in the quantum mechanical fluctuations of the atomic population operator are presented here; also the loss of coherence due to Rayleigh scattering can be analyzed using a perturbative approach. In a many atom system, the collective radiation effects are estimated employing a model in which a one-dimensional field is propagating through a pencil-shaped atomic medium. Maxwell-Bloch calculations are used to find an expression for the average scattered field amplitude; however, to study entanglement one requires a mixed semiclassical-quantized approach in which the driving field-atom interaction is treated semiclassically while the scattered field is treated quantum mechanically (by taking into account the interaction of the atoms with the quantum vacuum).

### A. One atom

The positive frequency part of the field (evaluated at position  $\mathbf{r}$ ) radiated by an atom (at the origin) can be expressed as (see Ref.[19])

$$\mathbf{E}^{(+)}(\mathbf{r}, t) \sim \frac{\omega_0^2}{r} \hat{\mathbf{r}} \times \mathbf{d}^{(-)}(t - \frac{r}{c}) \times \hat{\mathbf{r}} \quad (4)$$

where  $\mathbf{d}^{(-)}(t - \frac{r}{c})$  is the atomic dipole moment operator at a retarded time. In terms of atomic operators  $\mathbf{d}^{(-)}(t) = \sum_{m_g, m_e} \langle g, m_g | \mathbf{d} | e, m_e \rangle \sigma_{-}(m_e, m_g; t)$ . An average can be performed and the emitted field is found to be associated with a nonvanishing average atomic dipole moment. To simplify the calculations we assume that there are two different time scales; one for the evolution of the coherence between the ground and excited levels (characteristic time  $\gamma^{-1}$ ), and the other for ground state populations, which are driven by a pulse with a slowly varying amplitude (characteristic time  $T$ ). If  $\gamma T \gg 1$ , one can solve the Bloch equations for the coherences and excited state populations at a given time  $t$  taking a 'frozen' value of the Rabi frequency  $\chi(t)$  and ground state populations  $\rho_{g, -1/2; g, -1/2}(t)$  and  $\rho_{g, 1/2; g, 1/2}(t)$  at that particular time. A perturbative approach is valid in the limit  $\frac{\chi(t)^2}{\Delta^2 + (\gamma_e/2)^2} \ll 1$  for all times  $t$ , where  $\chi(t)$  is the Rabi frequency,  $\Delta$  is the detuning of the laser field from the resonant atomic frequency and  $\gamma$  represents the decay rate of the upper state manifold [20]. Expressions for the scattered field components expectation values (in the incident field direction  $z$ ) in terms of the radiating ground-excited state coherences are obtained as

$$\begin{aligned}
\langle E_x^{(+)}(z, t) \rangle &\sim [\rho_{g,-1/2;e,1/2}(t - z/c) + \rho_{g,1/2;e,-1/2}(t - z/c)] \\
\langle E_y^{(+)}(z, t) \rangle &\sim [\rho_{g,-1/2;e,1/2}(t - z/c) - \rho_{g,1/2;e,-1/2}(t - z/c)]
\end{aligned} \tag{5}$$

With adiabatic elimination of the excited state amplitudes, the coherences can be expressed in terms of the ground state populations only as

$$\begin{aligned}
\rho_{g,-1/2;e,1/2}(t) &= \frac{-\chi(t)/\sqrt{3}}{\Delta - i\frac{\gamma}{2}} \rho_{g,-1/2;g,-1/2}(t) \\
\rho_{g,1/2;e,-1/2}(t) &= \frac{-\chi(t)/\sqrt{3}}{\Delta - i\frac{\gamma}{2}} \rho_{g,1/2;g,1/2}(t)
\end{aligned} \tag{6}$$

and the fields become

$$\begin{aligned}
\langle E_x^{(+)}(z, t) \rangle &\sim \chi(t) \{ \rho_{g,1/2;g,1/2}(t - z/c) + \rho_{g,-1/2;g,-1/2}(t - z/c) \} \\
\langle E_y^{(+)}(z, t) \rangle &\sim \chi(t) \{ \rho_{g,1/2;g,1/2}(t - z/c) - \rho_{g,-1/2;g,-1/2}(t - z/c) \}
\end{aligned} \tag{7}$$

The  $y$  part is the Faraday rotated field and is seen to be produced (at least on average) by the population imbalance between the ground substates. However, the scattered field intensity (for both  $x$  and  $y$  polarized scattered fields) is isotropic and reflects only the variation of the driving field envelope  $I_{x,y}(z, t) \sim \langle E_{x,y}^{(-)}(z, t) E_{x,y}^{(+)}(z, t) \rangle \sim |\chi(t)|^2$ . We conclude that, when one measures the ratio of field amplitude  $\langle E_y^{(+)}(z, t) \rangle$  to the driving field amplitude, the noise affecting its detection (given by the ratio of  $I_y(z, t)$  to  $|\chi(t)|^2$ ) is always larger than the maximum signal. This problem can be overcome by making use of an ensemble of atoms where the signal scales as  $N_a^2$  while the noise scales only as  $N_a$ .

A final observation that we make and which will be useful when the decoherence effects of the spontaneous decay will be taken into account is that the ground state coherence decays at a rate proportional to the driving field intensity. The parameter describing the total loss of coherence during the time the laser pulse is on is the time integrated one atom spontaneous emission rate:

$$C_{spon}^2 = \gamma \int_0^T dt \frac{|\chi(t)|^2}{\Delta^2} \tag{8}$$



## B. Maxwell-Bloch approach

Having observed that the polarization rotation can be explained as a single atom effect, we calculate the response of a collection of such Faraday active atoms to an incident classical field. A pencil-shaped medium (length  $L$ ) with transverse area  $A$  and density  $n = N_a/AL$  is considered. Solving the Bloch equations for the atomic polarizability induced by the external field and substituting this back into the Maxwell equations, we obtain the induced amplitude and phase changes of the original field. The main result of Appendix A is that the Faraday rotation is given by

$$\phi = \frac{n |p|^2 \Omega L}{\hbar \Delta c \epsilon_0} [\rho_{g,1/2;g,1/2}(t) - \rho_{g,-1/2;g,-1/2}(t)] = \frac{2p^2 \Omega}{\hbar \Delta c \epsilon_0 A} \langle S_z \rangle \quad (9)$$

and is proportional to the average population imbalance  $\langle S_z \rangle$  of the entire ensemble. However, this analysis provides an expression for the average effect only. Notice that for a whole class of initial states exhibiting different population fluctuations but with the same vanishing expectation value of  $S_z$ , a vanishing signal is obtained. This comes from ignoring the interaction with the vacuum field or, equivalently, by treating the scattered field as classical.

## C. Source field approach

When the vacuum modes are taken into account, a mixed semiclassical-quantized field approach can be used in which the atomic dipole is driven by a classical field and radiates a source field in the quantized vacuum. The source field components in the forward direction at position  $z$  are obtained as (see Appendix B)

$$\begin{aligned} E_x^{(+)}(z, t) &= \frac{-inL\Omega |p|^2}{2\hbar\Delta\epsilon_0 c} E_0(0, t - z/c) e^{-i\Omega(t-z/c)} \\ E_y^{(+)}(z, t) &= \frac{nL\Omega |p|^2}{2\hbar\Delta\epsilon_0 c} E_0(0, t - z/c) e^{-i\Omega(t-z/c)} \frac{S_z}{S} \end{aligned} \quad (10)$$

The  $y$  part (unlike the  $x$  part) shows a dependence on a collective spin operator of the ensemble. A Faraday rotation operator can be defined as

$$\hat{\phi} = 2E_y^{(+)}(z, t) e^{i\Omega(t-z/c)} / E_0(z/c, t) = \frac{nL\Omega |p|^2}{\hbar\Delta\epsilon_0 c} \frac{S_z}{S} \quad (11)$$

having variance

$$\Delta\hat{\phi} = \frac{n|p|^2 \Omega L \Delta S_z}{\hbar \Delta c \varepsilon_0 S} \quad (12)$$

The mean value of the rotation angle is, as expected, the same as predicted in the previous subsection. The usefulness of the new result is that atomic states that exhibit different fluctuations give rise to source fields with different quantum properties. The detection of such a field can therefore provide information about collective atomic fluctuations in the ensemble.

#### IV. EFFECTIVE INTERACTION HAMILTONIAN

A simple form for the evolution generator of the system atoms-forward scattered field can be obtained by neglecting the coupling of atoms to any other field modes. In other words, spontaneous emission is neglected and the system follows a deterministic evolution described by an effective Hamiltonian that is derived below. Spontaneous emission can be introduced phenomenologically and will produce a degradation of the measurement-induced entanglement.

##### A. Effective Hamiltonian in the absence of decay

As seen in the previous section, the field operator (in the Heisenberg picture) for the  $y$  polarized mode is proportional to the atomic operator  $S_z$ . That suggests that entanglement between the field and atoms is present; however, a wave function (or density matrix) approach is needed to quantitatively estimate the extent of this entanglement. Reducing this situation to the problem of a driven quantized dipole radiating into the vacuum modes, one can find an effective interaction Hamiltonian (for derivation, see Appendix C). With the observation (justified in Appendix C) that the evolution of the  $x$  polarized mode is decoupled from the evolution of the atomic state and of the  $y$  polarized mode, the evolution operator generated by the Hamiltonian is

$$U(T) = e^{-iC(c_y^\dagger - c_y)S_z}, \quad (13)$$

where the interaction parameter  $C = \left[ \frac{3}{16\pi^2} \left( \frac{\lambda^2}{A} \right) \left( \gamma \int_0^T dt \frac{|x(t)|^2}{\Delta^2} \right) \right]^{1/2}$ . The operator  $c_y$  is the annihilation operator for the source field, that, when applied to the vacuum, creates a

quasimonochromatic  $y$  polarized one photon pulse of energy  $\hbar\Omega$  and duration  $T$ , propagating in the positive  $z$  direction.

The state vector for the atom-field system is given by

$$\begin{aligned}
|\Psi(T)\rangle &= U(T) |S_x = S\rangle \otimes |0\rangle_y = \sum_{M=-S}^S A(S, M) \exp[-iC(c_y^\dagger - c_y)S_z] |0\rangle_y |S, M\rangle = \quad (14) \\
&= \sum_{M=-S}^S A(S, M) \exp[-iC(c_y^\dagger - c_y)M] |0\rangle_y |S, M\rangle \\
&= \sum_{M=-S}^S A(S, M) \left( |-iCM\rangle_y^{coh} \right) \otimes |S, M\rangle = \\
&= \sum_{M=-S}^S A(S, M) e^{-(CM)^2/2} \left( \sum_{n_y=0}^{\infty} \frac{(-iCM)^{n_y}}{\sqrt{n_y!}} |n_y\rangle_y \right) \otimes |S, M\rangle
\end{aligned}$$

The evolution generator is a displacement operator whose amplitude is an operator rather than the usual c-number. As such it corresponds to a superposition of coherent states, each of which is associated with one of the eigenstates of  $S_z$ . It is evident from Eq.(14) that the participation of each atomic state to the final amplitude of the scattered field is governed by the distribution of the atomic fluctuations in the initial atomic state, namely the binomial coefficient  $A(S, M)$ . It is also seen that the detection of a number of photons in the proximity of one of the values  $C^2M^2$  indicates that one of the two atomic state with projections  $\pm M$  is responsible for scattering. However, since no phase information on the measured field is available, the states are indistinguishable, leading to a collapse of the atomic state onto a 'Schrodinger cat' state. An exception is the state with  $M = 0$ , which is simply connected with an outcome of zero scattered photons; such a 'null measurement' leads to the squeezing of the initial binomial distribution.

## B. Inclusion of spontaneous emission

In the above derivation, the coupling of atoms to field modes other than those belonging to the optical mode of interest (forward scattered field) is neglected. In consequence, even if the theory correctly describes the measurement strength that applies to a detection of the source field, it fails to account for the decay of the collective atomic coherence resulting from spontaneous emission into other vacuum modes. The coherence loss associated with  $C$  is only a fraction,  $\lambda^2/A$ , of the exact loss given by Eq. (8). The interpretation is

straightforward: in looking at the coupling to the forward scattered field only spontaneously emitted photons in a solid angle  $\lambda^2/A$  are considered.

The effect of spontaneous decay can be taken into account in two ways: either by limiting the number of spontaneously emitted photons per atom to a negligible value and proceed with an estimate of the optimum achievable entanglement, or by including the correct decay rate ( $C_{spon}^2$ ) in the expression for the evolution of the total mean spin of the sample (while the measurement process will still be described by a strength  $C^2$ ). The first approach (see Refs. [16, 23]), discussed in the following, requires a constant total spin of the sample (on the surface of the Bloch sphere), while in the second approach the optimization of the squeezing parameter as a function of  $C$  will provide the exact limitation (section VI).

For a pulse consisting of  $N_{ph}$  photons, off-resonantly interacting with the ensemble of atoms under consideration, the total loss is given by  $d_{off-res}N_{ph}$ , where  $d_{off-res}$  is the off-resonance optical depth of the sample. The condition of less than one photon loss per atom reads  $\eta = d_{off-res}N_{ph}/N_a < 1$  which leads to  $\eta = \left(\frac{d_{res}}{N_a}\right) \left(\frac{\gamma}{\Delta}\right)^2 N_{ph} < 1$ . Expressing  $C$  in terms of the number of incoming photons  $C \simeq \left(\frac{\gamma}{\Delta}\right) \left[\frac{\lambda^2}{A}\right] \sqrt{N_{ph}} = \frac{\gamma}{\Delta} \left(\frac{d_{res}}{N_a}\right) \sqrt{N_{ph}}$ , one can write  $C = \eta^{1/2} \sqrt{\frac{d_{res}}{N_a}}$  from which it is deduced that

$$C < \sqrt{\frac{d_{res}}{N_a}} \quad (15)$$

As a consequence,  $C$  is always less than unity; nevertheless, spin squeezing of order  $\sqrt{1/d_{res}}$  can still be achieved. For the regime in which the above condition is satisfied the effect of emission in directions other than into the mode subjected to detection can be safely omitted. The length of the mean spin stays approximately constant; in view of Eq. (3), the evolution of the orthogonal spin component variance as a result of the measurement is enough to describe squeezing. However, in the remaining sections, we'll use the second approach where the evolution of the system described by the effective Hamiltonian for any value of the measurement strength parameter is corrected by a fast decoherence of the average spin due to the inclusion of spontaneous decay.

## V. MEASUREMENT PROCESS

An indirect detection scheme can be imagined that closely resembles the case of the monitoring of a decaying field in a cavity [21]. The system is represented by the source field

plus atoms, while the photodetector is the environment. The monitoring of the photons that escape to the environment is made through the detection of photoelectrons. It is assumed that every absorbed photon produces one photoelectron which is registered with 100% efficiency. The detection process lasts for a time  $T$ , which is the time the source field interacts with the photodetector. The field passing through the detector is attenuated at a rate  $\lambda$  which gives a total attenuation about  $e^{-\lambda T}$  of the field during detection. With sufficiently large  $\lambda$ , all source field photons are detected, a condition that is equivalent to a 100% detection efficiency.

The formalism we use is one of continuous measurement theory. Our system of field and atoms loses photons at a rate  $\lambda$ . We then view the process as a piecewise deterministic process where periods of deterministic evolution are interrupted by sudden quantum jumps induced by the detection of a photon. The Lindblad jump operator is the one-mode effective annihilation operator for the source field  $c$ . The free evolution is determined by the non-hermitian operator:  $H_{nh} = -i\hbar\frac{\lambda}{2}c^\dagger c$  which generates an evolution operator  $U_d(t) = e^{-\frac{\lambda t}{2}c^\dagger c}$ . We are interested in the state of field and atoms after a detection time  $T_d$ ; therefore with notation  $1 - e^{-\lambda t} = \mu$ , the evolution operator becomes  $U_d(T_d) = (1 - \mu)^{c^\dagger c}$ .

The final state averaged over all detection histories (one history is a sequence of detection times inside the detection interval) that lead to a number of  $n_m$  detected photons (up to a phase factor) is

$$|\Psi_{n_m}\rangle = \frac{c^{n_m} U_d(T_d) |\Psi_{initial}\rangle}{\|c^{n_m} U_d(T_d) |\Psi_{initial}\rangle\|} \quad (16)$$

From Eq. (14), we obtain

$$|\Psi_{n_m}\rangle_{F+A} = \sum_{M=-S}^S \frac{A(S, M)(iCM)^{n_m} e^{-\mu(CM)^2/2}}{\sqrt{\sum_{X=-S}^S |A(S, M)|^2 (CM)^{2n_m} e^{-\mu(CM)^2}}} |S, M\rangle \otimes |iCM(1 - \mu)\rangle_y^{coh} \quad (17)$$

Two expected results are transparent here. Firstly, a field in a coherent state undergoing photodetection is left in a smaller (attenuated) amplitude coherent state. Secondly, when the condition of 100% detection efficiency is imposed by setting  $\mu = 1$ , the state of the field after measurement is the vacuum field. The decoupled entangled final state for the atoms is then given by

$$|\Psi_{n_m}\rangle_A = \sum_{M=-S}^S \frac{A(S, M)(iCM)^{n_m} e^{-(CM)^2/2}}{\sqrt{\sum_{X=-S}^S |A(S, M)|^2 (CM)^{2n_m} e^{-(CM)^2}}} |S, M\rangle \quad (18)$$

## VI. GENERATED ATOMIC ENTANGLEMENT

During the detection window (which is long enough to include the whole scattered field pulse) the detector registers a number of clicks which we regard as the outcome of the measurement. An outcome of  $n_m$  clicks indicates a particular set of atomic states that are most likely to have given rise to a field with  $n_m$  photons. Consequently, owing to the entanglement generated by the interaction, in response to the performed measurement, the atomic system is projected onto this set of states with properties that are analyzed below.

### A. Photon statistics

Equation (14) describes the field as a collection of coherent states with different amplitudes, each of them proportional to the corresponding atomic state that is responsible for its scattering. After tracing over the atomic states, a photon number distribution can be obtained, given by

$$P_{n_m}(C, S) = \sum_{N=-S}^S |A(S, N)|^2 e^{-(CN)^2} \frac{(CN)^{2n_m}}{n_m!} \quad (19)$$

Figure 2 is a plot of this function. Each peak is a superposition of two equal amplitude coherent states with opposite phase ( $\pm iCM$ ). The overall envelope is of binomial shape, that reflects the initial distribution of fluctuations in  $S_z$ . The locations of the peaks are  $0, C^2, C^2 2^2, \dots, C^2 M^2, \dots, C^2 S^2$ , while the width of an individual peak radiated by states  $\pm M$  is  $CM$ .

Although spontaneous emission will limit  $C$  to values less than unity, we consider large values to illustrate the entanglement mechanism. In the limit  $C \gg 1$ , it is seen that the overlap between consecutive peaks decreases since the separation between them scales as  $C^2$  while the width only scales as  $C$ . The consequence is that a measurement outcome effectively belongs to one peak only, indicating precisely the collective atomic state onto which the atoms are projected.

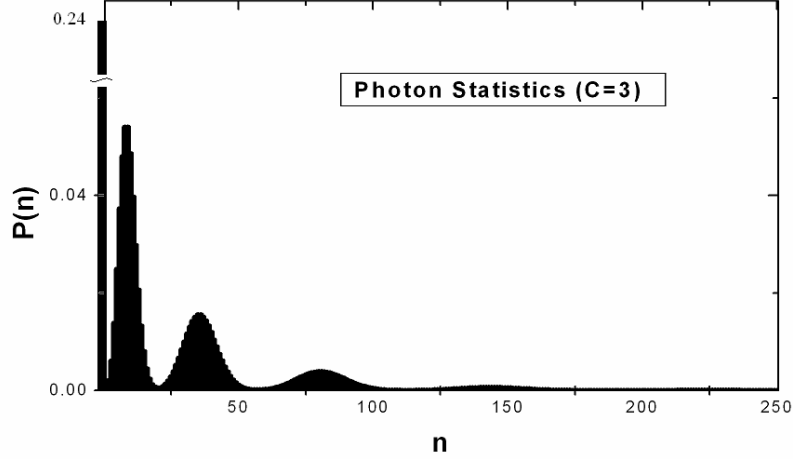


FIG. 2: Numerical plot of the photon number distribution for  $S = 10$  (20 atoms) and an intentionally exaggerated (for clarity of illustration) value of  $C = 3$ . There are 11 peaks located at 0, 9, 36, ..., 900 having halfwidths 0, 3, 6, ..30.

The expectation value and variance of the photon number operator are

$$\begin{aligned} \langle c_y^\dagger c_y \rangle &= C^2 \frac{N_a}{4} \\ \Delta(c_y^\dagger c_y) &= C^2 \sqrt{\frac{N_a}{4} \left\{ \frac{(N_a - 1)}{2} + \frac{1}{C^2} \right\}} \end{aligned} \quad (20)$$

In view of Eq. (15) the average number of scattered photons is seen to be actually limited by  $d_{res}$ , while a considerable amount of overlap among different peaks will make the task of separating the atomic states responsible for scattering very difficult.

### B. Null measurement. Spin squeezed states

From Eq. (18), the collective atomic state after a detection event with zero outcome is given by

$$|\Psi(T)\rangle_{\text{col}}^{n_m=0} = \frac{\sum_{N=-S}^S A(S, N) e^{-(CN)^2/2}}{\sqrt{\sum_{X=-S}^S |A(S, X)|^2 e^{-(CX)^2}}} |S, N\rangle \quad (21)$$

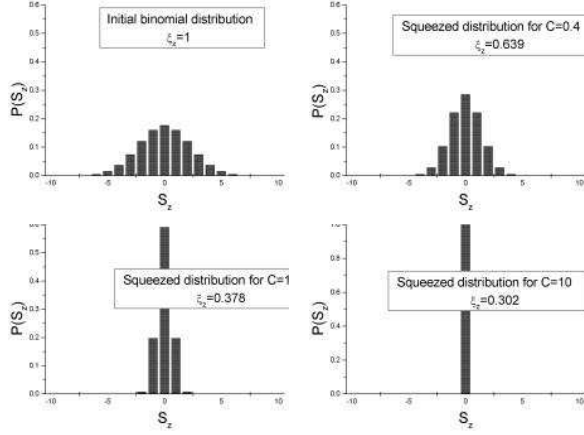


FIG. 3: The initial binomial distribution, for  $S = 10$ , is squeezed owing to the backaction of a measurement with null outcome. In the absence of decay to transverse modes, an ever better squeezing parameter is obtained that eventually tends to the Heisenberg limit when  $C \gg 1$ .

The atomic distribution function given by  $P_a^{(0)}(M) = |A(S, N)|^2 e^{-(CN)^2} / g(S, C)$  (where the normalization constant is given by  $g(S, C) = \left[ \sum_{N=-S}^S |A(S, N)|^2 e^{-(CN)^2} \right]^{1/2}$ ) is plotted above showing a squeezing from the initial distribution, owing to the exponentials in the numerator. An estimate of the width of the squeezed distribution can be found by using the Stirling approximation for the factorial of large numbers  $N! \simeq \sqrt{2\pi N} (N/e)^N$ . This value  $M_{1/e}$  is to be found as the point where the function decreases to  $1/e$  of the initial value  $P_a(0) = \frac{|A(S, 0)|^2}{g(S, C)} \simeq \frac{1}{\sqrt{\pi S} g(S, C)}$ . Noting that for the initial distribution  $M_{1/e}$  is  $\sqrt{S}$  and appreciable decrease of this value is expected (in the limit  $SC^2 \gg 1$ ),  $M_{1/e}$  is found to be given by  $C^{-1}$ ; remembering that  $C$  is proportional to the square root of the total number of scattered photons (in the forward direction), this is simply stating that the more photons are incident on the atomic sample, the sharper the distribution that can be obtained. This result is similar to the conclusions of Refs. [7, 8].

However, in order to estimate the optimal achievable squeezing, one has to also analyze the mean spin value. The  $y$  and  $z$  component expectation values vanish giving a mean spin pointing in the  $x$  direction:



$$\langle S_z \rangle = 0, \quad \langle S_y \rangle = 0, \quad \langle \mathbf{S} \rangle = \langle S_x \rangle \hat{x} \quad (22)$$

Ignoring spontaneous decay, for large values of  $C$  this is found to vanish asymptotically at a rate that is slower than the rate of information gathering (i.e. the rate of change in the variance of  $S_z$ ); the resulting spin squeezing approaches the Heisenberg limit. This is not true for the realistic case when decoherence is dominated by the much higher rate [from Eqs. (C12) and (8) it is derived that  $C_{spon}^2 \simeq C^2 \frac{N_a}{d_{res}}$ ]. The expression of the squeezing parameter in the presence of decay can be thus found (using Eqs.(3) and the previous derivations in the limit  $C \gg 1/\sqrt{S}$ )

$$\xi \simeq \sqrt{2S} \frac{(\frac{1}{C})}{S e^{-C_{spon}^2}} = \frac{\sqrt{2}}{\sqrt{S} C e^{-C^2 N_a / d_{res}}} \quad (23)$$

The function in the denominator reaches its maximum at  $C = \sqrt{d_{res}/(2N_a)} = \frac{1}{2} \sqrt{\frac{d_{res}}{S}}$ . This corresponds to a minimum squeezing

$$\xi_{\min} = \frac{2\sqrt{e}}{\sqrt{d_{res}}}$$

Notice that the best squeezing is obtained when the mean spin reaches a value around  $\sqrt{e}$  smaller than the initial maximal value; this corresponds to about one photon loss per atom. It should be added, that this being a probabilistic scheme, its success depends on the likelihood of a null measurement; this can be estimated (for large  $C$ ) as approaching a limiting value of  $\sqrt{\frac{2}{\pi N_a}}$  ( $\sqrt{\frac{1}{\pi S}}$ ). It should also be noted that collective effects have been neglected in estimating the decay of the mean spin resulting from spontaneous emission. Since one wishes to work in the limit  $n_a \lambda^2 L \gg 1$  (the limit usually associated with superradiance), one should, in principle, prove that the neglect of collective effects associated with spontaneous decay is justified.

### C. Non-null measurement. "Schrodinger Cat" states

The collapsed atomic state when  $n_m$  photons are detected is given by

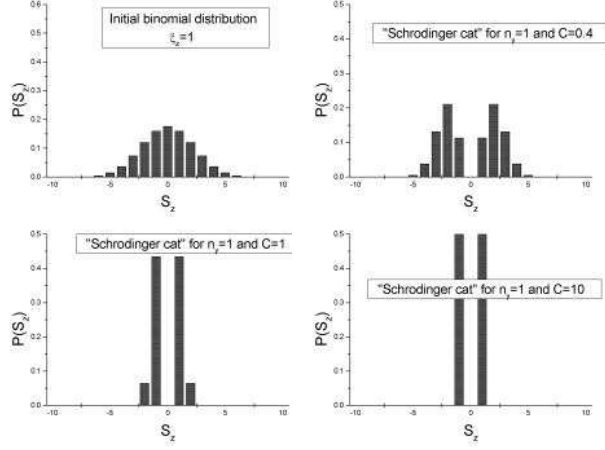


FIG. 4: An outcome of 1 photon projects the atomic system into a superposition of state which are probable to have given rise to a 1 photon field. With increasing  $C$  this superposition sharpens, turning eventually into a pure Schrodinger cat state.

$$|\Psi(T)\rangle_{\text{col}} = \frac{\sum_{N=-S}^S A(S, N)(-iCN)^{n_m} e^{-(CN)^2/2}}{\sqrt{\sum_{X=-S}^S |A(S, X)|^2 (CX)^{2n_m} e^{-(CX)^2}}} |S, N\rangle \quad (24)$$

and gives an atomic probability distribution  $P_a^{(n_m)}(M) = |A(S, M)|^2 (CM)^{2n_m} e^{-(CN)^2} / g(S, C, n_m)$  shown in Fig 4. The localization of the two peaks for given  $C, S$  and  $n_m$  can be made by finding the values at which the function  $|A(S, M)|^2 (CM)^{2n_m} e^{-(CN)^2}$  reaches its maxima. These are easily found to be given by  $\pm M_m = \pm \sqrt{n_m}/C$ . The sharpness of the peaks is found by using the same procedure as before in the null measurement case. Denoting the width with  $M_{1/e}^{(m)}$ , the condition of half maximum  $P_a^{(n_m)}(M_m + M_{1/e}^{(m)}) = P_a^{(n_m)}(M_m)/e$  yields the following equation :  $\left[1 + M_{1/e}^{(m)}/M_m\right]^{2n_m} = e^{C^2 \left[2M_m M_{1/e}^{(m)} + (M_{1/e}^{(m)})^2\right]}/e$ . A result which proves to set a fundamental distinction between our scheme and the one presented in Ref. [16] is numerically found, and that is that  $M_{1/e}^{(m)}/M_m < 1$  for any values of  $C$  and  $n_m$  which indicates that the arms of the cat are always distinguishable. This is due to the fact that the atomic state

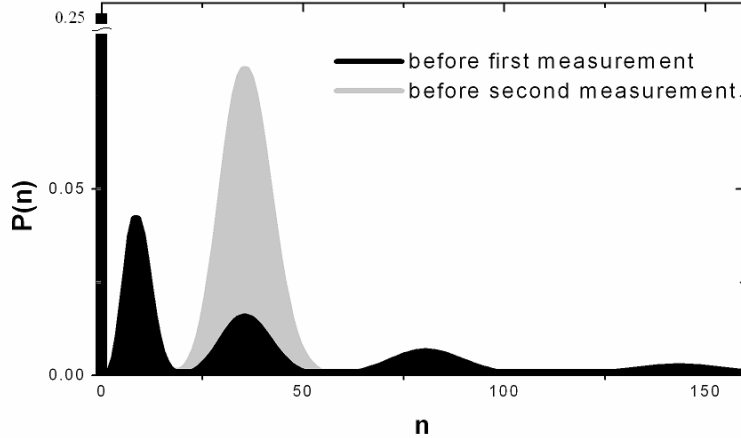


FIG. 5: After the first detection (with  $n_m = 30$ ) projected the atoms onto a Schrodinger cat state with  $M = \pm 2$ , the photon statistics before the second measurement changes accordingly (the gray plot has only one peak centered around  $n = 36$ ).

corresponding to a detection of any number of photons other than zero never contains the Dicke state with zero eigenvalue; therefore a clear separation between the left and right sides of the atomic population distribution. It should be also noted that, when  $C$  is around a value of  $\sqrt{d_{res}/S}$  (which, as seen in the previous section, leads to optimal squeezing for a null-measurement), the squeezing parameter for the cat state is approximately equal to  $\xi_x \simeq \sqrt{2SM_m}/S = \sqrt{\bar{n}_m}/\sqrt{SC^2} \simeq \sqrt{\bar{n}_m}/\sqrt{d_{res}}$ . Eq. (20) gives an average photon number (most probable measurement outcome)  $\bar{n}_m = d_{res}$ . The resulting squeezing parameter  $\xi_x$  can still be subunitary when the number of photons detected is smaller than the most probable value  $\bar{n}_m$ . This result increases the probability to obtain a squeezed state by adding the category of bimodal spin squeezed states to the single mode obtained with a zero detection outcome.

#### D. Subsequent measurements

After running the experiment once and obtaining a collapsed atomic collective state, this state can be probed by sending in a second pulse and performing the measurement once again. For well resolved peaks, the photon statistics before the second measurement contains only the peak that gave the outcome of the first measurement (as shown in Fig. 5).

We have performed computer simulations for  $N_a = 20$  atoms. The situation represented in Fig. 5 describes an ensemble of atoms initially prepared in a "Schrodinger cat" state. Taking  $C = 3$  and the measured number of photons  $n_m = 30$  (which is fairly close to the expectation value of  $C^2 N_a / 4 = 45$  photons and belongs to the second peak, centered at 36), after the photodetection process the atomic state is projected onto a "cat" state with  $S_z$  projections  $\pm 2$ . Sending another light pulse through the ensemble, the expected statistics is modified. A single peak representing a sum of two coherent states scattered by the states  $\pm 2$  survives. A next measurement is most likely only to sharpen the initially prepared "cat" state.

For realistic, small values of  $C$ , the peaks cannot be well resolved; therefore, after a single measurement, the resulting cat state will not be very sharp. However, following the procedure described in Ref.[6] where an initially coherent cavity field is projected onto a Fock state, the smallness of  $C$  can be compensated by continuing to send pulses and detect the source field until a reasonably good sharpness of the resulting atomic state is achieved (ideally a projection onto a superposition of Dicke states).

### E. Role of detection efficiency

Less than 100% efficient detection degrades the quality of squeezing and robustness of the Schrodinger cat states. Two different approaches have been used to include the effect of the undetected photons on the collapsed atomic state. In the first one, one assumes a subunitary efficiency parameter ( $\mu < 1$ ) while in the second one, an imaginary beam splitter (with transmissivity  $T$ ) is placed in the way of the incoming field diverting part of the light, while the rest undergoes perfect detection ( $\mu = 1$ ). Both calculations yield the same result if  $\mu = T$ . Using the first approach, the final state of the total system atoms-scattered field is given by Eq. (17). The non-vacuum field states after detection represent the part of the field that escaped detection due to inefficiency. A trace over these states needs to be performed and the density matrix elements of the final atomic state are given by

$$\rho_{MN}(\mu) = \frac{A^*(S, N)A(S, M) \{ \alpha_N^{*n_m} \alpha_M^{n_m} e^{(1-\mu)\alpha_N^* \alpha_M} \} e^{-(|\alpha_M|^2 + |\alpha_N|^2)/2}}{\sum_{X=-S}^S |A(S, X)|^2 |\alpha_X|^{2n_m} e^{-\mu|\alpha_X|^2}} \quad (25)$$

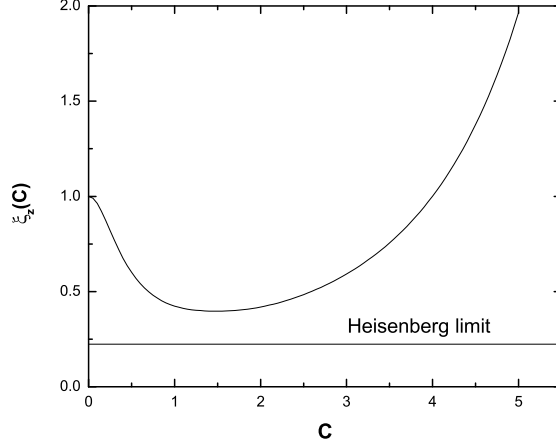


FIG. 6: For  $N_a = 20$  and 85% detection efficiency, optimal squeezing is reached around  $C = 2$ . Thereafter, an increase in the driving field strength only deteriorates the squeezing parameter.

with the notation  $\alpha_M = -iCM$ . For the spin squeezing case, when the detection outcome is zero, the variance of the population operator is given by

$$(\Delta S_z)^2 = \sum_{M=-S}^S M^2 \frac{|A(S, M)|^2 e^{-\mu|\alpha_M|^2}}{\sum_{X=-S}^S |A(S, X)|^2 e^{-\mu|\alpha_X|^2}} \quad (26)$$

showing a reduction at a slower rate than in the perfect detection case. It can be shown that, for  $C \gg 1$ , the mean spin nevertheless tends to zero at the same rate as in the perfect detection case. In consequence, the spin squeezing parameter doesn't tend to the Heisenberg limit with increasing field strength, but rather reaches a local minimum, tending to infinity afterwards (see Fig.6). The reason is that the decoherence due to the continuous scattering of photons on the ensemble continues at the same rate no matter what the detection efficiency is, while the information gathering is slower for smaller efficiency leading to a slower reduction in  $(\Delta S_z)^2$ . The value of  $C$  at which the squeezing reaches a minimum is estimated to be in direct relation to the detection efficiency

$$C \simeq \frac{1}{\sqrt{1-\mu}} \quad (27)$$

The Schrodinger cat state with arms at  $\pm M_m$  is even more sensitive to the detector inef-

iciency. Although the height of the two symmetric peaks  $[\rho_{\pm M; \pm N}(\mu)]$  is mostly insensitive to the efficiency, the coherence between them disappears as soon as

$$C \gtrsim \frac{1}{M_m \sqrt{1 - \mu}}$$

## VII. CONCLUSIONS

A source field approach to the interaction of a Faraday active atomic medium with an  $x$  polarized classical laser pulse has been considered here. The Faraday effect is responsible for the rotation of the field polarization and thus the redistribution of initially  $x$  polarized photons into the  $y$  polarized mode. The photon occupation number of this mode is seen to reflect the atomic population operator fluctuations; this leads to the observation that a measurement of the photon number operator provides a means for information acquisition on the collective atomic quantum state. The key point in this analysis is that, due to the use of pulsed classical fields rather than continuous quantized fields, the interaction is not a QND one. This scheme rather falls into the category of EPR measurements, where the interaction between subparts of the system has ceased before the actual measurement takes place. A "discrete" detection formalism is used that allows to describe the collapse of the system wave function conditioned on an outcome of  $n_m$  clicks at the photodetector. This is not much different from the procedure followed in Ref. [8] where the entanglement of two macroscopic atomic samples is produced by sending photons one by one through both media and updating the collective atomic wave function after each detection.

Our results are related to those of Ref. [13], when the measurement time window is matched to our pulse duration  $T$ . The main difference is that our choice of ignoring the phase information of the Faraday rotation (through the use of an unbalanced detection scheme) leads to the preparation of a superposition of symmetric squeezed states rather than a single squeezed state shifted to the left or right of the origin.

## VIII. ACKNOWLEDGEMENTS

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## IX. APPENDIX A: MAXWELL-BLOCH APPROACH

The incident field is an  $x$  polarized laser pulse with a slowly varying envelope  $E_0(z, t)$ , frequency  $\Omega$  and wave number  $k_0$ , propagating in the  $z$  direction. In a circular polarization basis with  $\hat{\epsilon}_+ = -\frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})$  and  $\hat{\epsilon}_- = -\hat{\epsilon}_+^*$ , it can be written as a superposition of a right and left circularly polarized fields

$$\mathbf{E}(z, t) = -\left\{\frac{1}{2\sqrt{2}}E_0(z, t)e^{ik_0z}e^{-i\Omega t}\hat{\epsilon}_+ + cc\right\} + \left\{\frac{1}{2\sqrt{2}}E_0(z, t)e^{ik_0z}e^{-i\Omega t}\hat{\epsilon}_- + cc\right\} \quad (\text{A1})$$

We proceed by first finding the phase change in the  $\sigma_+$  polarized light, which can induce transitions between  $|g, -1/2\rangle$  and  $|e, 1/2\rangle$ . The equations of motion for the density matrix elements can be solved readily to obtain

$$\rho_{g,-1/2;e,1/2}(z, t) = \frac{\Lambda(z, t)}{\Delta}\rho_{g,-1/2;g,-1/2}(z, t)e^{-i\Omega t} \quad (\text{A2})$$

where

$$\Lambda(z, t) = \frac{1}{2\sqrt{2}\hbar}E_0(z, t)e^{ik_0z}p \quad (\text{A3})$$

with  $p = e\langle g, -1/2 | r | e, 1/2 \rangle$  representing the dipole moment element of the transition. The substitution of the polarization of the medium

$P(z, t) = np\rho_{g,-1/2;e,1/2}(z, t) + cc$  into the Maxwell-Bloch equation for the phase shift

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\phi_+(z, t) = -\frac{k_0}{2\epsilon_0}\frac{\text{Re}(P(z, t))}{E_0(z, t)} \quad (\text{A4})$$

gives, in the steady state regime (assuming that the field depletion is negligible) a phase shift for the  $\sigma_+$  polarized light equal to

$$\phi_+ = -\frac{np^2\Omega L}{\hbar\Delta c\epsilon_0}\rho_{g,-1/2;g,-1/2} \quad (\text{A5})$$

For the  $\sigma_-$  component of the field, an identical calculation yields

$\phi_- = -\frac{np^2\Omega L}{\hbar\Delta c\epsilon_0}\rho_{g,1/2;g,1/2}$ . When the atoms are initially only in the two ground sublevels, the Faraday rotation angle is

$$\phi = \phi_+ - \phi_- = \frac{np^2\Omega L}{\hbar\Delta c\epsilon_0}[\rho_{g,1/2;g,1/2} - \rho_{g,-1/2;g,-1/2}] = \frac{2p^2\Omega}{\hbar\Delta c\epsilon_0 A}\langle S_z \rangle \quad (\text{A6})$$

## X. APPENDIX B: SOURCE-FIELD APPROACH

Here we look at the same medium as a collection of independent driven atoms confined in a pencil-shaped volume, that radiate a phased-matched forward field. The  $N$  atoms located at fixed positions  $\mathbf{R}_j$ , are far from the observer's position  $\mathbf{r}$  such that  $k|\mathbf{r} - \mathbf{R}_j| \gg 1$ . The positive frequency field amplitude operator at position  $\mathbf{r}$  is quantized in an infinite volume [with notation  $E_q(\omega_k) = \left(\frac{\hbar\omega_k}{2\varepsilon_0(2\pi)^3}\right)^{1/2}$ ]

$$\mathbf{E}^{(+)}(\mathbf{r}) = i \sum_{\lambda} \int d^3k E_q(\omega_k) a_{\lambda}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \hat{\epsilon}_{\lambda}(\mathbf{k}) \quad (\text{B1})$$

where the continuous field operators obey the usual commutations  $[a_{\lambda}^{\dagger}(\mathbf{k}), a_{\lambda'}(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}')\delta_{\lambda\lambda'}$ . The atoms in an infinitesimal volume  $\delta V(Z) = A n \delta Z$  respond the same way to the external driving field in the approximation that the transverse profile of the laser beam is constant over the cross-sectional area  $A$ . The components of a dimensionless,  $Z$  dependent atomic operator over the  $\delta V(Z)$  volume (containing  $\delta N_a(Z)$  atoms) can be defined as outlined in Ref. [22]

$$\sigma_{\alpha}(Z) = \lim_{\delta Z \rightarrow 0} \frac{1}{\delta N_a(Z)} \sum_{Z_j \in \delta V(Z)} \sigma_{\alpha}(Z_j) = \frac{1}{nA} \left( \lim_{\delta Z \rightarrow 0} \frac{1}{\delta Z} \sum_{Z_j \in \delta V(Z)} \sigma_{\alpha}(Z_j) \right) \quad (\text{B2})$$

which obey the following commutation relations  $[\sigma_x(Z), \sigma_y(Z)] = \frac{2}{nA} \sigma_z(Z) \delta(Z - Z')$ . The total spin of the sample with length  $L$  is defined as  $\mathbf{S} = \frac{nA}{2} \int_0^L \boldsymbol{\sigma}(Z) dZ$  and satisfies angular momentum commutation relations  $[S_x, S_y] = iS_z$ . The interaction part of the Hamiltonian (in the Schrodinger picture) can be written

$$\begin{aligned} H &= H_{CF-A} + H_{QF-A} \quad (\text{B3}) \\ H_{CF-A}(t) &= -\hbar \sum_{j=1}^{N_a} \sum_{m_g, m_e} \left[ \chi(m_e, m_g, z_j, t) e^{ik_0 Z_j} \sigma_+^{(j)}(m_e, m_g; Z_j) e^{-i\Omega t} + h.c. \right] \\ H_{QF-A} &= \hbar \sum_{j=1}^{N_a} \sum_{m_g, m_e} \sum_{\lambda} \int d^3k \left[ g_{\lambda}^*(m_e, m_g, \mathbf{k}) a_{\lambda}(\mathbf{k}) \sigma_+^{(j)}(m_e, m_g; Z_j) e^{i\mathbf{k}\cdot\mathbf{R}_j} + h.c. \right] \end{aligned}$$

with the notation  $\Delta = \omega - \Omega$ ,  $\Delta_k = \omega_k - \Omega$ ,  $\chi(m_e, m_g, Z_j, t) = \frac{E_0(Z_j, t) \langle e, m_e | \hat{\mathbf{x}} \cdot \mathbf{d} | g, m_g \rangle}{2\hbar}$  (classical field-atoms coupling strength) and  $g_{\lambda}(m_e, m_g, \mathbf{k}) = -\frac{i}{\hbar} E_q(\omega_k) \langle g, m_g | \hat{\epsilon}_{\lambda}^*(\mathbf{k}) \cdot \mathbf{d} | e, m_e \rangle$  (quantum vacuum-atoms coupling strength). It is convenient now to switch to the Heisenberg



picture and describe the time evolution of time dependent operators as generated by the Heisenberg picture Hamiltonian. The equation of motion for the  $a_\lambda^{HP}(\mathbf{k}, t)$ , after formal integration, is

$$a_\lambda^{HP}(\mathbf{k}, t) = a_\lambda^{HP}(\mathbf{k}, 0) e^{-i\omega_k t} - i \sum_{j=1}^{N_a} \sum_{m_g, m_e} g_\lambda(m_g, m_g, \mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{R}_j} e^{-i\omega_k t} \int_0^t \sigma_-^{(j)HP}(m_e, m_g; Z_j, t') e^{i\omega_k t'} dt' \quad (\text{B4})$$

Neglecting the free part (which doesn't contribute to expectation values) and transforming the summation over atoms into an integral, one can express the field as

$$\mathbf{E}^{(+)}(\mathbf{r}, t) = \frac{-i}{\hbar} \sum_{\lambda} \int d^3k \sum_{j=1}^{N_a} \sum_{m_g, m_e} g_\lambda(m_g, m_g, \mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{R}_j} e^{-i\omega_k t} e^{i\mathbf{k} \cdot \mathbf{r}} \left\{ \int_0^t \sigma_-^{(j)HP}(m_e, m_g; Z_j, t') e^{i\omega_k t'} dt' \right\} \hat{\epsilon}_\lambda(\mathbf{k}) \quad (\text{B5})$$

Using the Heisenberg equations of motion one can adiabatically eliminate the ground-excited level coherences. Owing to the excitation scheme [shown in Fig. 1(b)], they are connected only to diagonal elements  $\sigma_g^{(j)}(m_g, m_g; t)$  (ground state populations)

$$\sigma_-^{(j)HP}(m_e, m_g; Z_j, t) = -\frac{\chi(m_g, m_e; t)}{\Delta} \sigma_g^{(j)HP}(m_g, m_g; t) e^{ik_0 Z_j} e^{-i\Omega t'} \quad (\text{B6})$$

Note that, the population operators do not depend on position (due to the homogeneity of the medium) or on time (due to the far-off resonance regime in which we work) and therefore can be replaced by their corresponding operators in the Schrodinger picture  $\sigma_g^{(j)}(m_g, m_g)$ . Substituting this result back in Eq. (B5) and summing over polarizations with  $\hat{\epsilon}_1(\mathbf{k}) = \hat{\mathbf{x}}$  and  $\hat{\epsilon}_2(\mathbf{k}) = \hat{\mathbf{y}}$ , and

$$\sum_{\lambda} g_\lambda(m_g, m_g, \mathbf{k}) \hat{\epsilon}_\lambda(\mathbf{k}) = -\frac{i}{\hbar} E_q(\omega_k) \langle g, m_g | \mathbf{d} | e, m_e \rangle \quad (\text{B7})$$

we arrive at the electric field operator

$$\mathbf{E}^{(+)}(\mathbf{r}, t) = \frac{i}{\hbar \Delta} \sum_{m_g, m_e} \int d^3k E_q^2(\omega_k) e^{-i\omega_k t} e^{i\mathbf{k} \cdot \mathbf{r}} \left\{ \int_0^t dt' \chi(m_g, m_e; t') \left[ \sum_{j=1}^{N_a} e^{-i(k_x X_j + k_y Y_j)} e^{i(k_0 - k_z) Z_j} \sigma_g^{(j)}(m_g, m_g) \right] e^{i\omega_k t'} e^{-i\Omega t'} \right\} \langle g, m_g | \mathbf{d} | e, m_e \rangle \quad (\text{B8})$$

The sum over the atoms can be transformed into an integral. In the limit of large transverse area  $A \gg \lambda^2$ , the integral over  $X$  and  $Y$  is performed, yielding

$$\sum_{j=1}^{N_a} e^{-i(k_x X_j + k_y Y_j)} e^{i(k_0 - k_z) Z_j} \sigma_g^{(j)}(m_g, m_g) = 4\pi^2 n \delta(k_x) \delta(k_y) \int_0^L dZ e^{i(k_0 - k_z) Z} \sigma_g(m_g, m_g) \quad (\text{B9})$$

Grouping all the terms that depend on  $\omega_k$ , and evaluating the slowly varying term  $E_q^2(\omega_k)$  at  $\Omega$ , the integral over the field modes becomes  $\int d\omega_k e^{-i\omega_k(t-t'-z/c+Z/c)} = \delta(t-t'-z/c+Z/c)$ . Evaluating the  $t'$  integral now by replacing  $t' = t - z/c + Z/c$ , the  $e^{ik_0 Z}$  term is canceled, which allows us to perform the integral over  $Z$  and introduce collective population operators  $S_g(m_g, m_g) = \frac{nA}{2} \int_0^L dZ \sigma_g^{(j)}(m_g, m_g)$ . The total field becomes

$$\mathbf{E}^{(+)\text{HP}}(\mathbf{r}, t) = \frac{8i\pi^2 E_q^2(\Omega)}{\hbar \Delta c A} \left\{ \sum_{m_g, m_e} \chi(m_g, m_e; t) S_g(m_g, m_g) \langle g, m_g | \mathbf{d} | e, m_e \rangle \right\} e^{-i\Omega(t-z/c)} \quad (\text{B10})$$

Separating the field into  $x$  and  $y$  components, and noting that  $S_g(1/2, 1/2) + S_g(-1/2, -1/2) = S$  and  $S_g(1/2, 1/2) - S_g(-1/2, -1/2) = S_z$  one obtains the following expressions for the field at position  $z$

$$E_x^{(+)\text{HP}}(z, t) = \frac{-inL\Omega |p|^2}{2\hbar \Delta \varepsilon_0 c} E_0(0, t - z/c) e^{-i\Omega(t-z/c)} \quad (\text{B11})$$

$$E_y^{(+)\text{HP}}(z, t) = \left\{ \frac{nL\Omega |p|^2}{2\hbar \Delta \varepsilon_0 c S} E_0(0, t - z/c) e^{-i\Omega(t-z/c)} \right\} S_z$$

A Faraday rotation operator can be defined as:

$$\hat{\phi} = 2E_y^{(+)}(z, t) e^{i\Omega(t-z/c)} / E_0(z/c, t) = \frac{nL\Omega |p|^2}{\hbar \Delta \varepsilon_0 c} \frac{S_z}{S} \quad (\text{B12})$$

whose expectation value coincides with the result obtained using a Maxwell-Bloch approach.

## XI. APPENDIX C: DERIVATION OF THE EFFECTIVE HAMILTONIAN

An effective Hamiltonian that describes the generation of a quantized source field by the driven atomic system can be derived from  $H_{QF-A}$  in Eq. (B3). The steps that

are necessary in this procedure are discussed here. First, using the previously defined continuous atomic operators, one can transform the sum over atoms into an integral  $\sum_{j=1}^{N_a} \sigma_a^j e^{i\mathbf{k}\cdot\mathbf{R}_j} \rightarrow n_a \int dX dY dZ \sigma(Z) e^{i\mathbf{k}\cdot\mathbf{R}} = n_a \int_0^L dZ \sigma(Z) \int_A dX dY e^{i\mathbf{k}\cdot\mathbf{R}}$ . An integration over the  $x$  and  $y$  components of  $\mathbf{k}$  followed by one over  $X$  and  $Y$  allows us to define an effective one-mode continuous field operator that describes photons propagating in the  $z$  direction with polarization  $\lambda$  and transverse spatial extend  $A$

$$d_\lambda(k_z) = \frac{1}{2\pi\sqrt{A}} \int dX dY \int dk_x dk_y a_\lambda(\mathbf{k}) e^{ik_x X} e^{ik_y Y} \quad (\text{C1})$$

The commutation relations are  $[d_\lambda(k_z), d_{\lambda'}^\dagger(k'_z)] = \delta(k_z - k'_z) \delta_{\lambda\lambda'}$ . With the newly defined field operator and continuous atomic operators replaced in Eq. (B3) the expression for the interaction part of the Hamiltonian becomes

$$H_{QF-A} = 2\pi\hbar n A^{1/2} \sum_{m_g, m_e, \lambda} \int dk_z \int_0^L dZ \left\{ g_\lambda(m_g, m_e; k_z) d_\lambda^\dagger(k_z) \sigma_-(m_e, m_g; Z) e^{-ik_z Z} + h.c \right\} \quad (\text{C2})$$

Using the equivalent of Eq. (B6) for continuous operators, one can replace coherences by population operators (in the Heisenberg picture)

$$\sigma_-^{HP}(m_e, m_g; Z, t) = -\frac{\chi(m_g, m_e; t)}{\Delta} \sigma_g^{HP}(m_g, m_g; t) e^{ik_z Z} e^{-i\Omega t} \quad (\text{C3})$$

The Heisenberg picture Hamiltonian is found in terms of population operators. Negligibly small Langevin fluctuations have been ignored here since their correlations are smaller than the term driving the coherence by a factor of  $\gamma/\Delta$ . With a transformation back to the Schrodinger picture, an expression for the effective Hamiltonian is found. With removal of the free evolution of field and atoms, and neglecting the light shift of the lower levels, an interaction picture Hamiltonian that will govern the evolution of the system is found to be

$$H_{QF-A}^{IP} = 2\pi\hbar n A^{1/2} \sum_{m_g, m_e, \lambda} \int dk_z \int_0^L dZ \left\{ \frac{g_\lambda(m_g, m_e; k_z) \chi(m_g, m_e; t)}{\Delta} d_\lambda^\dagger(k_z) \sigma_g(m_g, m_g) e^{i(k_0 - k_z)Z} e^{i(\omega_k - \Omega)t} + h.c \right\} \quad (\text{C4})$$

Since the regime we work in is the low saturation limit where the absorption of the field is negligible, the population operators are not changed by the spatial dependence of the field amplitude and therefore  $Z$  independent. Noting that the integral over  $k_z$  varies rapidly outside a small interval around  $k_0$ ,  $g_\lambda(m_g, m_e; k_z)$  (which varies slowly as  $\sqrt{k_z}$ ) can

be evaluated at  $k_0$ . The time dependence of the coupling strength includes  $E(0, t)$  which is the slowly varying envelope of the field with Fourier components  $E(\omega_k - \Omega)$  that are nonzero only in a small interval centered at the carrier frequency  $(\omega_k - \Omega) \ll 1/T$ . Consequently, the allowed interval for the wave vectors is  $(k_0 - k_z) \ll 1/cT$ . In the limit of  $L \ll cT$  (all atoms see the same field amplitude at one instant in time), it is implied that  $(k_0 - k_z) Z \ll Z/cT < L/cT \ll 1$ . This condition allows us to set the spatial dependence  $e^{i(k_0 - k_z)Z}$  to 1 and evaluate  $\chi(m_g, m_e, Z, t)$  at  $Z = 0$ . The integral over the sample length introduces the collective spin operator. That means that the forward scattered field couples only to the symmetric atomic mode through the lowering and raising spin operators  $S_+, S_-$ . The sum over  $m_g, m_e$  can be performed now for each polarization ( $x$  and  $y$ ) with the result for the simplified Hamiltonian:

$$H_{QF-A}^{IP} = \left\{ \frac{2i\pi |p|^2}{\hbar\Delta\sqrt{A}} E_q(\Omega) E(0, t) \left( \int dk_z d_x^\dagger(k_z) e^{i(\omega_k - \Omega)t} \right) S + h.c \right\} + \left\{ \frac{2\pi |p|^2}{\hbar\Delta\sqrt{A}} E_q(\Omega) E(0, t) \left( \int dk_z d_y^\dagger(k_z) e^{i(\omega_k - \Omega)t} \right) S_z + h.c \right\} \quad (C5)$$

Notice that the first term in the rhs commutes with both  $y$  mode field operators and atomic operators, which means that the  $x$  polarized field is decoupled from the rest of the system. Equivalently, one can say that only the  $y$  part carries information about the quantum state of the atomic ensemble. The  $x$  part can thus be discarded, and setting  $B = \frac{2\pi|p|^2}{\hbar^2\Delta\sqrt{A}} E_q(\Omega)$  the  $y$  part takes the form:

$$(H_{QF-A}^{IP})_y = \hbar B E(0, t) \left( \int dk_z d_y^\dagger(k_z) e^{i(\omega_k - \Omega)t} \right) S_z + h.c \quad (C6)$$

With the observation that

$$\left[ (H_{QF-A}^{IP})_y(t), (H_{QF-A}^{IP})_y(t') \right] = 0 \quad (C7)$$

the evolution operator can be written in a simple form:

$$U(T) = \exp \left[ -\frac{i}{\hbar} \int_0^T (H_{QF-A}^{IP})_y(t) dt \right] \quad (C8)$$

The time integral brings the Fourier components of the incident pulse field envelope  $\int_0^T dt e^{i(\omega_k - \Omega)t} E(0, t) \simeq E(\omega_k - \Omega)$  giving

$$U(T) = \exp \left[ -i \left( B \left\{ \int dk_z E(\omega_k - \Omega) d_y^\dagger(k_z) \right\} S_z + h.c. \right) \right] \quad (\text{C9})$$

The integral over the continuous creation operators can be represented by an effective one-photon creation operator with carrier frequency  $\Omega$  and duration  $(c\Delta k)^{-1} \simeq T$  and obeying the commutation relation  $[c_y, c_y^\dagger] = 1$

$$c_y^\dagger = \frac{c^{1/2}}{\sqrt{\int_0^T dt |E(t)|^2}} \int dk_z E(\omega_k - \Omega) d_y^\dagger(k_z) \quad (\text{C10})$$

which leads to a simple form for the evolution operator

$$U(T) = \exp[-iC(c_y^\dagger - c_y)S_z] \quad (\text{C11})$$

with  $C = \frac{B}{c^{1/2}} \sqrt{\int dk_z |E(\omega_k - \Omega)|^2} = \frac{2\pi|p|^2 E_q(\Omega)}{\hbar^2 \Delta \sqrt{Ac^{1/2}}} \sqrt{\int_0^T dt |E(t)|^2}$ . In terms of one atom total spontaneous emission loss during the time  $T$  of the applied laser pulse [see Eq. 8]  $C$  can also be expressed as

$$C = \left[ \frac{3}{16\pi^2} \left( \frac{\lambda^2}{A} \right) \left( \gamma \int_0^T dt \frac{|\chi(t)|^2}{\Delta^2} \right) \right]^{1/2} = \left[ \frac{3}{16\pi^2} \left( \frac{\lambda^2}{A} \right) C_{\text{spont}} \right]^{1/2} \quad (\text{C12})$$

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